Oscillatory Motion
Oscillatory Motion

A time sequence showing one complete cycle for the vibration of a mass on a spring.
**MASS-SPRING SYSTEM**

Unstrained length of the spring

\[ F = -kx \quad \text{RESTORING FORCE} \]

**Simple Harmonic Motion**

- Stretching force
- Restoring force
- Velocity
- Restoring force
- Velocity
- Equilibrium position

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**EQN OF MOTION:**
\[ m\ddot{x} = F \]
\[ ma = F \]
\[ ma = -kx \]
\[ m\frac{d^2x}{dt^2} = -kx \]

**TRY:**
\[ x(t) = e^{wt} \]
\[ \frac{dx}{dt} = we^{wt} \]
\[ \frac{d^2x}{dt^2} = w^2e^{wt} \]

**SUBSTITUTE:**
\[ m(\omega^2 e^{wt}) = ? \]
\[ = -k(e^{wt}) \]
\[ mL\omega^2 = ? \]
\[ mL\omega^2 = -k \]

**IMPOSSIBLE FOR REAL \( \omega \)**

**Try:**
\[ x = \cos(\omega t + \phi) \]
\[ \frac{dx}{dt} = -\omega \sin(\omega t + \phi) \]
\[ \frac{d^2x}{dt^2} = -\omega^2 \cos(\omega t + \phi) \]

**SUBST.:**
\[ m[-\omega^2 \cos(\omega t + \phi)] = -k[\cos(\omega t + \phi)] \]
\[ mL\omega^2 = -k \Rightarrow \text{OK if } |\omega^2| = \frac{k}{m} \]

**Adding Arbitrary Constant \( A \):**
\[ x(t) = A \cos(\omega t + \phi) \]
\[ \omega = \sqrt{\frac{k}{m}} \]

**Interpretation of Constants:**
- \( A \) = Amplitude = \( \max |x| \)
- \( T \) = Period = Time of One Cycle

**Figure 11.2**

**Vibrating Spring Motion**

\[ x(t+T) = x(t) : \cos[\omega(t+T)+\phi] = \cos(\omega t + \phi) \]

\( \omega \rightarrow \text{Angular Frequency} \)
\[ \omega = \frac{2\pi}{T} \]
\[ 2\pi \cdot \frac{1}{T} \]

\( f \rightarrow \text{Frequency} \)
\[ f = \frac{1}{T} \]
\[ \omega = 2\pi f \]
If $\phi = 0$, then
$$x = A \cos \omega t$$

**Plot of $x(t)$:**
- $x = A \sin \frac{2\pi t}{T}$

Simple Harmonic Motion

- $\cos$/$\sin$ curve
\[ x(t) = A \cos \left( \frac{2\pi t}{T} + \phi \right) \]

\[ \omega = \frac{2\pi}{T} \]

**Amplitude and Period**

**Phase**

\[ \phi \]

**Part of Phase**

**Constant Phase**

**Current**

\[ I(t) = A \cos (\omega t + \phi) \]

\[ \omega = \sqrt{\frac{K}{m}} \]

\[ \omega = \frac{1}{\sqrt{LC}} \]
T52 (Figure 13-5) Velocity in simple harmonic motion

\[ v = \frac{dx}{dt} = \frac{d}{dt} \left[ A \cos(\omega t + \phi) \right] \]
\[ = -\omega A \sin(\omega t + \phi) \]
\[ v_{\text{max}} = \omega A \]
\[ v = -v_{\text{max}} \sin(\omega t + \phi) \]

Position of mass

Time \( T = \text{period} \)

0

Simple Harmonic Motion

\[ \phi = 0 : x = A \cos \omega t \]

\[ \text{x largest, } v \text{ zero} \]

\[ \text{x zero, } v \text{ largest} \]
FIGURE 11-2  Force on, and velocity of, mass at different positions of its oscillation.

- **(a)**: When $x = 0$, $v = 0$.
- **(b)**: When $x = -A$, $v = -v_0$ (max. in negative direction).
- **(c)**: When $x = A$, $v = 0$ (max. in positive direction).
- **(d)**: When $x = 0$, $v = 0$.

Acceleration $a = \frac{dv}{dt} = -\omega^2 x$,

$x(t) = A \cos(\omega t)$,

$v(t) = -\omega A \sin(\omega t)$,

$a(t) = -\omega^2 A \cos(\omega t)$.
Displacement, velocity, and acceleration in simple harmonic motion

Figure 11.4

\[ \tau = ma \]
Dependence of SHM on phase $\phi$

$$x(t) = A \cos(\omega t + \phi)$$

$x = A \cos \omega t$

$x = A \sin \omega t$

$x = -A \cos \omega t$

$x = -A \sin \omega t$

$A \cos(\omega t + \frac{3\pi}{2}) = x = A \sin \omega t$

Same period, phase constant; different amplitude:

Same amplitude, phase constant; different period:

Same amplitude, period; different phase constant;

Dependence of simple harmonic motion on the phase constant.
T50 (Figure 13-2) Phase in simple harmonic motion

\[ \omega = \sqrt{\frac{k}{m}} \; ; \; \text{how to find } A, \theta ? \]

They depend on initial conditions \( v(0) = v_0, \; x(0) = x_0 \)

\[ \text{Advance in time by } \frac{\delta}{\omega} \]

\[ A \sin (\omega t + \delta) = A \sin \left[ \omega \left( t + \frac{\delta}{\omega} \right) \right] \]

\[ x(t) = A \cos(\omega t + \phi) \]

\[ v(t) = -\omega A \sin(\omega t + \phi) \]

\[ x_0 = A \cos \phi \]

\[ v_0 = -\omega A \sin \phi \]

\[ \text{Solve for } A, \phi \text{ in terms of } x_0, v_0 : \]

\[ \cos^2 \phi + \sin^2 \phi \delta = \frac{x_0^2}{A^2} + \frac{v_0^2}{\omega^2 A^2} \]

\[ 1 = \frac{1}{A^2} \left( x_0^2 + \frac{v_0^2}{\omega^2 A^2} \right) \]

\[ A = \sqrt{x_0^2 + \left( \frac{v_0}{\omega A} \right)^2} \; \; \text{"A" found} \]
Release at rest: \( v_0 = 0 \); \( x_0 \neq 0 \)

\[
A = \sqrt{x_0^2 + v_0^2 / \omega^2} = |x_0|
\]

\[
\tan \omega = -\frac{v_0}{\omega x_0} = 0 \Rightarrow \omega = 0; \quad x = A \cos \omega t
\]

\[
v = -\omega A \sin \omega t
\]

\[
a = -\omega^2 A \cos \omega t
\]
15. A 0.5 kg mass attached to a spring of force constant 8 N/m vibrates with simple harmonic motion with an amplitude of 10 cm. Calculate (a) the maximum value of the speed and acceleration, (b) the speed and acceleration when the mass is x = 6 cm from the equilibrium position, and (c) the time it takes the mass to move from x = 0 to x = 8 cm.

Solution
(a) \( \omega = \sqrt{\frac{k}{m}} = \frac{8 \text{ N/m}}{0.5 \text{ kg}} = 4 \frac{\text{rad}}{\text{s}} \)
Therefore, position is given by \( x = A \sin(4t) \). From this we find that
\[
\begin{align*}
   v &= 40 \text{ cm/s} \sin(4t) \\
   a &= 160 \text{ cm/s}^2 \sin(4t)
\end{align*}
\]
(b) \( t = \frac{1}{4} \sin^{-1} \left( \frac{x}{10} \right) \)
When \( x = 6 \text{ cm} \), \( t = 0.161 \text{ s} \) and we find
\[
\begin{align*}
   v &= 40 \text{ cm/s} \sin(0.161) = 33 \text{ cm/s} \\
   a &= 160 \text{ cm/s}^2 \sin(0.161) = -99 \text{ cm/s}^2
\end{align*}
\]
(c) Using \( t = \frac{1}{4} \sin^{-1} \left( \frac{x}{10} \right) \)
When \( x = 0 \), \( t = 0 \) and when \( x = 8 \text{ cm} \), \( t = 0.232 \text{ s} \). Therefore \( \Delta t = 0.232 \text{ s} \)

\[
\begin{align*}
   x &= 10 \sin 4t \\
   \frac{x}{10} &= \sin 4t \\
   \sin^{-1} \left( \frac{x}{10} \right) &= 4t \\
   \frac{x}{10} &= \sin \left( \frac{x}{10} \right)
\end{align*}
\]
**Energy of the Oscillator:**

\[ E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \]

- **Potential energy function for a mass on a spring**
- **Table:**
  - Column headers: \( x \), \( e \), \( K \), \( U \)
  - Entries:
    - \( x \): \( 0 \), \( -A \), \( A \)
    - \( e \): \( 0 \), \( 0 \), \( 0 \)
    - \( K \): \( 0 \), \( \frac{1}{2}kA^2 \), \( \frac{1}{2}kA^2 \)
    - \( U \): \( 0 \), \( \frac{1}{2}kA^2 \), \( \frac{1}{2}kA^2 \)

- **Graph:**
  - Points: \( P_1 \), \( P_2 \)
  - Equation: \( U = \frac{1}{2}kx^2 \)
  - **Equations:**
    - \( x = A \cos (\omega t + \phi) \)
    - \( u = -\omega A \sin (\omega t + \phi) \)
    - \( E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \)
    - \( \frac{1}{2}kA^2 \sin^2 (\phi) + \frac{1}{2}kA^2 \cos^2 (\phi) = \frac{1}{2}kA^2 \)
**SHIFTED CTR OF VIBRATIONS:**

\[ x = x' + \frac{mg}{k} \]

\[ x' = A \cos(\omega t + \delta) \]
ENERGY OF THE OSCILLATOR:

\[ E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \]

\[ U = \frac{1}{2}kx^2 \]

\[ E = \frac{1}{2}m\omega^2x^2 = 2\pi^2m\omega^2 \]

\[ E = K + U = \frac{1}{2}kA^2 \]

DIRECT PROOF OF MECH. ENERGY CONSERVATION

\[ E = K + U = \frac{1}{2}kA^2 \]
Figure 14: Motion of a block on a horizontal surface colliding with a light spring.

(a) Initial position, \( x = 0 \), \( v = 0 \), \( E = \frac{1}{2} m v_i^2 \).

(b) \( x = 0 \), \( v > 0 \), \( E = \frac{1}{2} m v_i^2 + \frac{1}{2} kx^2 \).

(c) \( v = 0 \), \( x \neq 0 \), \( E = \frac{1}{2} k A_n^2 \).

(d) \( v_i > 0 \), \( x_{nm} \), \( E = \frac{1}{2} m v_{\text{max}}^2 \).

\( \text{Note: } A_n \text{ is the maximum displacement.} \)
23. A particle executes simple harmonic motion with an amplitude of 3.0 cm. At what displacement from the midpoint of its motion will its speed equal one half of its maximum speed? 

Solution

From energy considerations, $v^2 = \omega^2 x^2 = \omega^2 A^2$.

$v_{\text{max}} = \omega A$ and $v = \frac{v_{\text{max}}}{\sqrt{2}} = \frac{\omega A}{\sqrt{2}}$ so

$\frac{1}{2} \omega^2 A^2 + \frac{1}{2} \omega^2 x^2 = \frac{1}{2} \omega^2 A^2$

From this we find $x^2 = \frac{3A^2}{4}$ and $x = \pm \frac{\sqrt{3}A}{2} = \pm \frac{3\sqrt{3}}{2} = \pm 2.60 \text{ cm}$ where $A = 3.0 \text{ cm}$.

$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

$E = \frac{1}{2}kA^2$
T53 (Figure 13-9) The simple pendulum

(a)

Arc length $s = \theta$

(b)

$mg \cos \theta$

$mg \sin \theta$

Figure 13.29
\[ \frac{d^2\phi}{dt^2} = -\omega^2 \phi, \text{ denoting } \omega^2 = \frac{g}{L} \]

SAME EQN. AS FOR SHO \Rightarrow

\[ \Theta = \Theta_0 \cos(\omega t + \phi), \quad \omega = \sqrt{\frac{g}{L}} \]

\[ \Theta_0 \text{ small.} \]

\[ \text{PERIOD } T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}} \]

FOR SMALL OSCILLATIONS.
PHYSICAL PENDULUM:
\[ I \ddot{\theta} = 0 \]
\[ I \frac{d^2 \theta}{dt^2} = -mgd \sin \theta \]

**Assume** \( \theta \approx \theta \):

\[ \sin \theta \approx \theta \]

\[ I \frac{d^2 \theta}{dt^2} = -mgd \theta \]

\[ \omega = \sqrt{\frac{mgd}{I}} \]

\[ \frac{d^2 \theta}{dt^2} = -\omega^2 \theta - \text{AGAIN SKM!} \]

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{mgd}} \]

\[ I = I_{cm} + md^2 \]
CHAPTER 10

33. A physical pendulum in the form of a planar body exhibits simple harmonic motion with a frequency of 1.5 Hz. If the pendulum has a mass of 2.2 kg and the pivot is located 0.35 m from the center of mass, determine the moment of inertia of the pendulum.

Solution

\[ f = 1.5 \text{ Hz}, \quad d = 0.35 \text{ m}, \quad \text{and} \quad m = 2.2 \text{ kg} \]

\[ T = \frac{k}{f} \quad T = 2\pi \sqrt{\frac{I}{mgd}} \quad I = \frac{4\pi^2 mgd}{T^2} \]

\[ I = \frac{T^2 mgd}{4\pi^2} = \frac{(2.2 \text{ kg})(9.8 \text{ m/s}^2)(0.35 \text{ m})}{(1.5 \text{ s}^{-1})^2(4\pi^2)} \]

\[ I = 8.50 \times 10^{-2} \text{ kg m}^2 \]
**FIGURE 11-5** Analysis of simple harmonic motion as a side view (b) of circular motion (a).

The text states:

*SHM is represented as a projection of circular motion (v = cos Ï€) onto the diameter.*

**FIGURE 10.17**

The figure illustrates the displacement x of the shadow being the projection of the radius a onto the x-axis. The equation given is:

\[ x = A \cos \theta = A \cos \omega t \]
\[ u_T = Aw \]
\[ v_x = -u_T \sin \theta \]
\[ v = -Aw \sin \theta \]
\[ \theta = \frac{dx}{dt} \]

**Velocity** \( v_x \) of the shadow is the \( x \)-component of the velocity \( u_T \) of the ball on the reference circle.

\[ a = -a_c \cos \theta = -\frac{v^2}{A} \cos \theta = -\frac{w^2 A \cos \theta}{\frac{dx}{dt}} \]

**FIGURE 10.19**

\[ v = wA \]

**FIGURE 10.22**
T51 (Figure 13-4) Uniform circular motion projects to simple harmonic motion
DAMPED OSCILLATIONS

\[ A e^{-\frac{bt}{2m}} \]

(a)

Undamped \((b = 0)\)

Small \(b\)

Large \(b\)

(b)

DAMPING OF SIMPLE HARMONIC MOTION BY A DRAG FORCE

\[ \vec{F} = -\vec{B} \dot{\vec{y}} \]

Viscous Medium

\[ m \ddot{y} = \vec{F}_{el} + \vec{F} \]

(1) \[ m \frac{d^2x}{dt^2} = -kx - B \frac{dx}{dt} \]

For small \(B\):

\[ x(t) = A e^{-\frac{bt}{2m}} \cos (\omega' t + \phi) \]

\[ \omega' = \sqrt{\frac{k}{m} - \left(\frac{B}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{\omega_d}{2m}\right)^2} \]

Verify by substitution into (1)

Displacement as a function of time for a damped oscillator.
No Damping: 
\( b = 0 \)

Exponential Decay of Amplitude: 
\( \frac{b}{2m} < \omega_n \)

Underdamped Case with small \( b \)

Underdamped Case with large \( b \)

Overdamped Case: 
\( \frac{b}{2m} > \omega_n \)

Critically Damped Case: 
\( b = \omega_n \frac{2m}{2m} \)

**NO OSCILLATIONS**
FORCED OSCILLATIONS: DAMPED OSCILLATOR DRIVEN BY AN EXTERNAL HARMONIC FORCE

\[ F = F_0 \cos \omega t \]

\[ m \ddot{x} = \frac{\dot{x}}{L} + \frac{\dot{\zeta}}{L} + \frac{F_0}{L} \]

\[ m \frac{d^2x}{dt^2} = -kx - \delta \frac{dx}{dt} + \frac{F_0}{m} \cos \omega t \]

Undamped
Small Damping
Large Damping

Natural \( \omega_0 = \sqrt{\frac{k}{m}} \)

Steady State:
\[ x(\text{large } \epsilon) = A \cos(\omega t + \phi) \]

\[ A(\omega) = \frac{F_0}{\sqrt{(\omega^2 - \omega_0^2)^2 + (\delta \omega)^2}} \]
Figure 14-19

Sudden increase in amplitude when driving frequency \( \omega \) near natural frequency \( \omega_0 \): resonance

\[
A(\omega) = \begin{cases} 
\frac{x_m}{b/2m} & \text{for small damping} \\
\frac{x_m}{b/2m} \left(1 + \frac{\omega^2 - \omega_0^2}{\omega_c^2}ight) & \text{for large damping}
\end{cases}
\]

\( b = 50 \text{ g/s} \) (smallest damping)

\( b = 70 \text{ g/s} \)

\( b = 140 \text{ g/s} \)

Figure 14-21

Power absorbed by a driven oscillator:

\[
P_{av} = \frac{F \omega_0}{2} \frac{1}{\omega_c} 
\]

Resonance curves for power absorbed by a driven oscillator

The smaller the damping, the sharper the resonance peak.

Width at half-maximum:

\[
\Delta \omega \approx \frac{2\omega_0}{b}
\]

Small damping, large \( Q \)

Large damping, small \( Q \)

Quality factor of the oscillator:

\[
Q = \frac{\omega_0}{\Delta \omega}
\]

measures sharpness of the resonance.