Solve 2.3, page 60: 2
Find the general solution of the DE.

\[
\frac{dy}{dx} + 2y = 0
\]

classify DE.
1st order, linear, homogeneous, ODE.

First, let's separate variables.

\[
\frac{dy}{dx} = -2y
\]
\[
\frac{dy}{y} = -2dx
\]

Note: We must assume \( y \neq 0 \) so that we can divide by zero.

Integrate.
\[
\int \frac{dy}{y} = \int -2dx
\]

\[
e^{\ln|y|} = -2x + C_1 \text{ where } C_1 \text{ is an arbitrary constant}
\]

\[
e^{\ln|y|} = e^{-2x+C_1}
\]

\[
|y| = e^{-2x+C_1} = e^{C_1}e^{-2x} = C_2e^{-2x}, \ C_2 \text{ is an arbitrary positive constant}
\]

\[
y = \pm C_2e^{-2x} = C_3e^{-2x}, \ C_3 \text{ is an arbitrary nonzero constant}
\]

check \( y = 0 \) by direct substitution.

\[
\Rightarrow \quad \frac{dy}{dx} = \frac{d(0)}{dx} = 0
\]
\[
\frac{dy}{dx} + 2y = 0 + (2)(0) = 0
\]

Thus, we the general solution

\[
y(x) = Ce^{-2x}, \text{ where } C = \text{arbitrary constant}
\]

Let's have SN solve the DE.
\[
\frac{dy}{dx} + 2y = 0, \text{ Exact solution is: } \{Ce^{-2x}\}
\]

Our answers agree.

\[
e^x
\]
Now, let's use an integrating factor.

\[ \frac{dy}{dx} + 2y = 0 \]

integrating factor for a first order, linear, ODE.

\[ e^{\int 2dx} = e^{2x} \]

Multiply each term of the DE by the integrating factor.

\[ e^{2x} \frac{dy}{dx} + 2ye^{2x} = 0e^{2x} \]

\[ \Rightarrow e^{2x} \frac{dy}{dx} + 2ye^{2x} = 0 \]

\[ \Rightarrow \frac{d(ye^{2x})}{dx} = 0 \]

That is, we should have the left side of the DE in the form

\[ \frac{d(\text{dependent variable} \times \text{integrating factor})}{dx} \]

Multiply each side by \( dx \) and then integrate.

\[ d(ye^{2x}) = 0dx \]

\[ \int d(ye^{2x}) = \int 0dx \]
\[ ye^{2x} = 0 + C \]
\[ ye^{2x} = C \]

\[ y = Ce^{-2x} \]

\[ \frac{dy}{dx} + 2y = 0 \]

Is this DE exact?

A DE of the form \( Mdx + Ndy = 0 \) is exact if
\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

\[ 2ydx + dy = 0 \]

Let \( M(x,y) = 2y \)
Let \( N(x,y) = 1 \)
\[ \frac{\partial M}{\partial y} = \frac{\partial (2y)}{\partial y} = 2 \]
\[ \frac{\partial N}{\partial x} = \frac{\partial (1)}{\partial x} = 0 \]

\[ 2 \neq 0 \Rightarrow \text{This DE is not exact.} \]

Is the DE multiplied by the integrating factor exact?

\[ e^{2x} \frac{dy}{dx} + 2ye^{2x} = 0 \]

\[ 2ye^{2x}dx + e^{2x}dy = 0 \]

Let \( M(x,y) = 2ye^{2x} \)
Let \( N(x,y) = e^{2x} \)
\[ \frac{\partial M}{\partial y} = \frac{\partial (2ye^{2x})}{\partial y} = 2e^{2x} \]
\[ \frac{\partial N}{\partial x} = \frac{\partial (e^{2x})}{\partial x} = 2e^{2x} \]
\[ 2e^{2x} = 2e^{2x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

Therefore, the DE is exact.
There exists a function \( f(x,y) \) with

\[
f = \int M \, dx = \int N \, dy
\]

\[
f = \int M \, dx = \int 2ye^{2x} \, dx = ye^{2x} + K(y)
\]

\[
f = \int N \, dy = \int e^{2x} \, dy = ye^{2x} + L(x)
\]

Equate the two forms of \( f \),

\[
f(x,y) = ye^{2x}
\]

The implicit form of the solution of an exact DE is \( f(x,y) = C \).

\[
\frac{ye^{2x} = C}{\Rightarrow y = Ce^{-2x}}
\]