MTH 291 - Differential Equations
Northern Virginia Community College
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Section 2.4 page 69: 31 (Differential Equations with Boundary Value Problems, 7 ed, Zill & Cullen)

Solve the given differential equation by finding, as in Example 4, an appropriate integrating factor.

\[(2y^2 + 3x)dx + 2xydy = 0\]

Let's check to see if the DE is exact.

Let \(M_1(x,y) = 2y^2 + 3x\)
Let \(N_1(x,y) = 2xy\)

\[
\frac{\partial M_1}{\partial y} = \frac{\partial (2y^2 + 3x)}{\partial y} = 4y
\]
\[
\frac{\partial N_1}{\partial x} = \frac{\partial (2xy)}{\partial x} = 2y
\]

\[4y \neq 2y\]

\[\Rightarrow \frac{\partial M_1}{\partial y} \neq \frac{\partial N_1}{\partial x}\]

\[\Rightarrow \text{the DE is not exact.}\]

We must find an integrating factor that will make the DE exact.

From page 67 of our textbook, we have the following results.

If \(\frac{M_y - N_x}{N}\) is a function of \(x\) alone, then an integrating factor for \(Mdx + Ndy = 0\) is

\[\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}\]

If \(\frac{N_x - M_y}{M}\) is a function of \(y\) alone, then an integrating factor for \(Mdx + Ndy = 0\) is

\[\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}\]

\[
\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{2y}{2xy} = \frac{1}{x}, \text{ which is a function of } x \text{ alone.}
\]

\[\Rightarrow \text{an integrating factor is } e^{\int \frac{M_y - N_x}{N} dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x\]
We can use $x$.

Let’s check the other case.

$$\frac{N_x - M_y}{M} = \frac{2y - 4y}{2y^2 + 3x} = -2 \frac{y}{2y^2 + 3x},$$ but this is not a function only of $y$, so this does not help.

Multiply each term of the DE by $x$.

$$x(2y^2 + 3x)dx + x2xydy = 0$$

$$\Rightarrow (2y^2 + 3x^2)dx + x^2ydy = 0$$

Let’s check to see that this DE is exact.

Let $M(x,y) = 2xy^2 + 3x^2$

Let $N(x,y) = 2x^2y$

$$\frac{\partial M}{\partial y} = \frac{\partial (2xy^2 + 3x^2)}{\partial y} = 4xy$$

$$\frac{\partial N}{\partial x} = \frac{\partial (2x^2y)}{\partial x} = 4xy$$

$$4xy = 4xy$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \text{the DE is exact.}$$

$$3f(x,y) = \int Mdx = \int Ndy$$

$$\int Mdx = \int (2xy^2 + 3x^2)dx = x^2(y^2 + x) = x^3 + x^2y^2 + K(y)$$

$$\int Ndy = \int (2x^2y)dy = x^2y^2 + L(x)$$

Equate the two forms of $f(x,y)$

$$f(x,y) = x^3y^2 + x^3$$

$K(y) = a \text{ constant}$

$L(x) = x^3$

The implicit solution of the DE is $f(x,y) = c$

$\therefore$ the solution is $x^3y^2 + x^3 = c$

Let’s see what happens when SN tries to solve this DE explicitly.

$$(2y^2 + 3x) + 2xy \frac{dy}{dx} = 0,$$ Exact solution is: $\left\{ \pm \sqrt{-x^3 - C_6}, -\frac{1}{2} \sqrt{-x^3 - C_6} \right\}$
Let's solve our implicit solution for $y$.

\[ x^2y^2 + x^3 = c, \text{ Solution is:} \]

\[ \begin{align*}
\emptyset & \quad \text{if } c = 0 \land x = 0 \\
\mathbb{C} & \quad \text{if } c \neq 0 \land x = 0 \\
\left\{ \pm \sqrt{c-x^3} : -\frac{1}{x} \sqrt{c-x^3} \right\} & \quad \text{if } x \neq 0
\end{align*} \]