MTH 291 - Differential Equations
Northern Virginia Community College
Extended Learning Institute
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Section 2.6 page p. 79: 5 (*Differential Equations with Boundary Value Problems, 7 ed, Zill & Cullen*)

Use a numerical solver and Euler's Method to obtain a four-decimal approximation of the indicated value.
First use \( h = 0.1 \) and then use \( h = 0.05 \)

\[
\begin{align*}
y' &= e^y \\
y(0) &= 0 \\
y(0.5) &= 0 \\
\frac{dy}{dx} &= e^y, \text{ Functions defined: } y \\
y(0) &= 0 \\
y(0.5) &= 0.40547
\end{align*}
\]

\( y(1) = 0.69315 \)
\( y(2) = 1.0986 \)
\( y(-1) \)
This gave a beep with no output.
When I tried to graph \( y \), SN froze and I had to close it.

I then figured out how fool SN.
I plotted \( y = 0 \) to obtain a figure box.
To eliminate the negative values of \( x \), I changed the view to \( 0 \leq x \leq 5, -5 \leq y \leq 5 \).
Then, I dragged the \( y \) from the Functions defined and dropped it into the figure box.
This produces part of the solution curve.

To see the graph below, you will have to open Plot Properties, delete item 2, that is, the numeric function \( y \). Then, you will have to delete ", Functions defined: \( y'' \) above, redo Compute - Solve ODE - Numeric, and drag the new \( y \) into the figure box.
Here is a screen shot from my Casio ClassPad.
It shows the direction field and the solution curve going through \((0, 0)\).

Next, we try to estimate \(y(0.5)\) using Euler’s method, with \(h = 0.1\)

Let \(f(x, y) = e^{-y}\)
We use the recursive formula \(y_{n+1} = y_n + hf(x_n, y_n)\).

\[x_0 = 0, \quad y_0 = 0\]
\[y_1 = y_0 + 0.1f(x_0, y_0) = 0 + 0.1(e^0) = 0 + 0.1(1) = 0.1\]

\[y_2 = y_1 + 0.1f(x_1, y_1) = 0.1 + 0.1(e^{-0.1}) = 0.19048\]

\[y_3 = y_2 + 0.1f(x_2, y_2) = (0.19048) + 0.1(e^{-0.19048}) = 0.27314\]
\[y_4 = y_3 + 0.1f(x_3, y_3) = (0.27314) + 0.1(e^{-0.27314}) = 0.37314\]
\[ y_5 = y_4 + 0.1 f(x_4, y_4) = (0.37314) + 0.1(e^{-0.37314}) = 0.44200 \]

We enter these values into the table below at each step.

\[ h = 0.1 \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.19048</td>
</tr>
<tr>
<td>3</td>
<td>0.3</td>
<td>0.27314</td>
</tr>
<tr>
<td>4</td>
<td>0.4</td>
<td>0.37314</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
<td>0.44200</td>
</tr>
</tbody>
</table>

\[ \therefore y'(0.5) \approx 0.44200 \]

We repeat the above process with \( h = 0.05 \).

We use the recursive formula \( y_{n+1} = y_n + hf(x_n, y_n) \).

\[ x_0 = y_0 = 0 \]
\[ y_1 = y_0 + 0.05 f(x_0, y_0) = 0 + 0.05(0) = 0 + 0.05(1) = 0.05 \]
\[ y_2 = y_1 + 0.05 f(x_1, y_1) = 0.05 + 0.05(e^{-0.05}) = 0.7561 \times 10^{-2} \]
\[ y_3 = y_2 + 0.05 f(x_2, y_2) = (0.7561 \times 10^{-2}) + 0.05(e^{-0.7561 \times 10^{-2}}) = 0.14291 \]
\[ y_4 = y_3 + 0.05 f(x_3, y_3) = (0.14291) + 0.05(e^{-0.14291}) = 0.19291 \]
\[ y_5 = y_4 + 0.05 f(x_4, y_4) = (0.19291) + 0.05(e^{-0.19291}) = 0.23414 \]
\[ y_6 = y_5 + 0.05 f(x_5, y_5) = (0.23414) + 0.05(e^{-0.23414}) = 0.2737 \]
\[ y_7 = y_6 + 0.05 f(x_6, y_6) = (0.2737) + 0.05(e^{-0.2737}) = 0.3173 \]
\[ y_8 = y_7 + 0.05 f(x_7, y_7) = (0.3173) + 0.05(e^{-0.3173}) = 0.34834 \]
\[ y_9 = y_8 + 0.05 f(x_8, y_8) = (0.34834) + 0.05(e^{-0.34834}) = 0.38363 \]
\[ y_{10} = y_9 + 0.05 f(x_9, y_9) = (0.38363) + 0.05(e^{-0.38363}) = 0.41770 \]

We enter these values into the table below at each step.

\[ h = 0.05 \]
\[
\begin{array}{ccc}
\ h \ \ n \quad x_n \\ 
0 & 0 & 0 \\
1 & 0.05 & 0.05 \\
2 & 0.10 & 9.7561 \times 10^{-2} \\
3 & 0.15 & 0.14291 \\
4 & 0.20 & 0.19291 \\
5 & 0.25 & 0.23414 \\
6 & 0.30 & 0.2737 \\
7 & 0.35 & 0.31173 \\
8 & 0.40 & 0.34834 \\
9 & 0.45 & 0.38363 \\
10 & 0.5 & 0.41770 \\
\end{array}
\]

\[ \therefore y(0.5) \approx 0.41770 \]

To check our work, let's solve the initial-value problem analytically and approximate the exact answers by decimals.

\[
y' = e^{-x}, \quad \text{Exact solution is: } y(x) = \ln(x + 1)
\]

\[ y(0) = 0 \]

\[ y(0.5) = \ln(0.5 + 1) \approx 0.40547 \]

This agrees with the numerical solver of SN.

We note that Euler's method gave a more accurate result with \( h = 0.05 \).

I disagree with the textbook answers of \( y_5 = 0.4198 \) and \( y_{10} = 0.4124 \).